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Tensors with icosahedral symmetry that are invariant under a certain wreath product: II

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Abstract. Previously we considered a problem arising in the theory of quasicrystals pointed out to us by Trebin. It amounts to the computation of a basis of a subspace of $\bigotimes^{2d}\mathbb{R}^3$, $d\in\mathbb{N}$ characterised by certain symmetries. We gave the solution for $d\leqslant 3$. Now we are able to describe such invariants for arbitrary d.

1. Introduction

Recall the mathematical problem (Kerber and Scharf 1987). It is well known that the icosahedral group is isomorphic to the direct product of the symmetric group in two letters and the alternating group in five letters: $I\simeq S_2\times A_5$. The corresponding symmetry operations can be realised by orthogonal representations of type $[1^2]\,\sharp\,[3,1^2]^{\pm}$ (see James and Kerber (1981) for notations). Consider the tensor space $(\mathbb{R}^3)^{\otimes^{2d}}$, $d\in\mathbb{N}^*$. The wreath product $S_2\wr S_d$ acts on this space by permuting the indices, if we identify it with the subgroup of S_{2d} generated by

$$(12),(13)\,(24),(135\ldots 2d-1)(246\ldots 2d).$$

Thus we obtain a representation of $S_2\wr S_d$. On the other hand we have the natural representation $([1^2]\,\sharp\,[3,1^2)^{\pm})^{\otimes^{2d}}$ of I on $(\mathbb{R}^3)^{\otimes^{2d}}$. As the automorphisms of I commute with those of $S_2\wr S_d$, we get a representation of $I\times S_2\wr S_d\simeq S_2\times A_5\times S_2\wr S_d$ on $(\mathbb{R}^3)^{\otimes^{2d}}$. The problem is to compute a basis of the space $(\mathbb{R}^3)^{\otimes^{2d}}_{S_2\times A_5\times S_2\wr S_d}$ of $S_2\times A_5\times S_2\wr S_d$-invariants for $d\in\mathbb{N}^*$. As we consider a direct product of groups, we have

$$(\mathbb{R}^3)^{\otimes^{2d}}_{S_2\times A_5\times S_2\wr S_d}=(((\mathbb{R}^3)^{\otimes^{2d}}_{S_2})_{S_2\wr S_d})_{A_5}.$$

But the group $S_2\times\{e_{A_5}\}\leqslant I$ acts as $\pm id$ on $\mathbb{R}^3$, whence it acts trivially on $(\mathbb{R}^3)^{\otimes^{2d}}$. We may therefore restrict attention to the group $A_5\times S_2\wr S_d$. It is easy to see that $(\mathbb{R}^3)^{\otimes^{2d}}_{S_2\wr S_d}$ is isomorphic to $\mathrm{Sym}_d(\mathrm{Sym}_2(\mathbb{R}^3))$ as vector spaces (we denote by $\mathrm{Sym}_i(-)$ the space of symmetric tensors of degree i). The isomorphism is intertwining, if we consider the representation of A_5 on $\mathrm{Sym}_d(\mathrm{Sym}_2(\mathbb{R}^3))$ induced by the representation of type $[3,1^2]^{\pm}$ on $\mathbb{R}^3$ and if we restrict the representation on the space of $S_2\wr S_d$-invariants to A_5. Note that we get a representation of type $([3,2]\oplus[1^5])\downarrow A_5$ on $\mathrm{Sym}_2(\mathbb{R}^3)$ both for $[3,1^2]^+$ and $[3,1^2]^-$. Note also that the problem stated above can be reformulated as follows. Describe the ($\mathbb{N}$-)graded algebra $\mathrm{Sym}(\mathrm{Sym}_2(\mathbb{R}^3))_{A_5}$. Each choice of a basis $(b_1,\ldots,b_6)$ of $\mathrm{Sym}_2(\mathbb{R}^3)$ gives a matrix representation of A_5. This matrix representation induces a representation on $\mathbb{R}[X_1,\ldots,X_6]$, the algebra of real polynomials in six

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indeterminates, such that the isomorphism of graded algebras—induced by $X_i \mapsto b_i$—is intertwining. Thus it suffices to restrict attention to $\mathbb{R}[X_1, \ldots, X_6]$, and we are in the setup of classical invariant theory. The problem is to describe

$$\mathbb{R}[X_1, \ldots, X_6]_{A_5}$$

the algebra of A_5-invariant polynomials in a suitable way. We shall do this in two ways:

(i) by giving a minimal system of homogeneous generators, i.e. a set of homogeneous polynomials generating this algebra which is of minimal length;

(ii) by providing a good polynomial basis, i.e. homogeneous polynomials

$$(q_1, \ldots, q_6; w_1, \ldots, w_{12})$$

such that

$$\mathbb{R}[X_1, \ldots, X_6]_{A_5} = \bigoplus_{i=1}^{12} \mathbb{R}[q_1, \ldots, q_6] w_i$$

the q_i being algebraically independent.

2. Choosing an appropriate basis

The calculations in Kerber and Scharf (1987) used the canonical basis of $\mathrm{Sym}_2(\mathbb{R}^3)$. But the geometry of the problem supplies a better one (see also Speiser 1923, p 132). Consider an icosahedron in $\mathbb{R}^3$ centred at the origin. The (six) lines through the origin and the vertices of the icosahedron are just the lines stabilised by the subgroups of A_5 isomorphic to the dihedral group D_{10}. Thus any $g \in A_5$ permutes the lines. An easy calculation gives the following proposition.

2.1. Proposition. If we take vectors $v_i (i = 1, \ldots, 6)$ of length one such that $\{\mathbb{R} \cdot v_i | i = 1, \ldots, 6\}$ is the collection of said lines, then the tensors

$$b_i := v_i \otimes v_i \qquad 1 \leqslant i \leqslant 6$$

form a basis of $\mathrm{Sym}_2(\mathbb{R}^3)$. Moreover, as we chose the representation on $\mathbb{R}^3$ to be orthogonal, this basis is permuted by the action of A_5.

Hence the intertwining mapping $X_i \mapsto b_i$ induces a permutation representation of A_5 on $\mathbb{R}[X_1, \ldots, X_6]$ which we shall consider next.

3. Permutation representations on $\mathbb{R}[X_1, \ldots, X_6]$

Consider the symmetric group S_6 in 6 letters. We have a natural representation of this group on $\mathbb{R}[X_1, \ldots, X_6]$ simply by permuting the X_i. In this case the theory of S_6-invariants is well known.

3.1. Theorem. The algebra of S_6-invariants is a polynomial algebra generated by six, algebraically independent, homogeneous polynomials

$$\mathbb{R}[X_1, \ldots, X_6]_{S_6} = \mathbb{R}[q_1, \ldots, q_6]$$

where we can choose the elementary symmetric functions or the power sums, for example.

The action of A_5 also permutes the X_i. Thus we may identify, for our purposes, (the representations are faithful!) A_5 with a subgroup of S_6. (It should be mentioned that this subgroup is not the 'usual' A_5 in S_6 or one of its conjugates: any element of order three is the product of two disjoint 3-cycles.)

First, information on the structure of A_5-invariants is given by the following theorem.

3.2. Theorem.

$$\mathbb{R}[X_1, \ldots, X_6]_{A_5} \supseteq \mathbb{R}[X_1, \ldots, X_6]_{S_6}.$$

Moreover, $\mathbb{R}[X_1, \ldots, X_6]_{A_5}$ is a free and finitely generated $\mathbb{R}[X_1, \ldots, X_6]_{S_6}$-module, i.e. there are homogeneous A_5-invariant polynomials $w_1, \ldots, w_r$ such that any A_5-invariant polynomial can uniquely be written as

$$\sum_i F_i \cdot w_i$$

where the F_i are polynomials in the q_i. (Hence $(q_1, \ldots, q_6; w_1, \ldots, w_r)$ is a good polynomial basis.) The number r and the degrees of the w_i are, up to permutations, uniquely determined. We have

$$r = \prod_i \deg(q_i)/|A_5| = 12$$

and the following table of degrees:

i	1	2	3	4	5	6	7	8	9	10	11	12
$\deg(w_i)$	0	3	5	6	6	7	8	9	9	10	12	15

(if $\deg(w_1) \leqslant \ldots \leqslant \deg(w_{12})$).

Proof. Let $R := \mathbb{R}[X_1, \ldots, X_6]$. The inclusion is obvious. First we explicitely prove that R_{A_5} is a finitely generated R_{S_6}-module (cf, for example, Springer 1977, lemmas 2.4.3, 4.1.2). Let $p \in R$. Then p is a zero of the monic polynomial $\Pi_{g \in S_6} (T - D(g)p) \in R_{S_6}[T]$. Hence the finitely-generated algebra R is finitely-generated as R_{S_6}-module (by the monomials $X_1^{a_1} \ldots X_6^{a_6}$, $a_i \leqslant |S_6|$, for example). As R_{S_6} is a finitely generated algebra, it is Noetherian by the Hilbert basis theorem, and, again by the Hilbert basis theorem, so is the R_{S_6}-module R. Hence the submodule R_{A_5} is finitely generated. It follows that $(q_1, \ldots, q_6)$ is a homogeneous system of parameters (Stanley 1979, p 481). Thus R_{A_5} is free as R_{S_6}-module (Stanley 1979, theorem 3.2). By Molien's formula (Stanley 1979, theorem 2.1), we are able to compute the Poincaré series $P_{A_5}(T)$ simply from the eigenvalues of the representing automorphisms of A_5 on $\bigoplus_i \mathbb{R}X_i$ (Conway *et al* 1985). If we choose a good polynomial basis $(q_1, \ldots, q_6; w_1, \ldots, w_r)$, then it is easy to see that the Poincaré series is equal to $\Sigma_j T^{\deg(w_j)}/\Pi_i (1 - T^{\deg(q_i)})$. Hence $\Sigma_j T^{\deg(w_j)} = P_{A_5}(T) \Pi_i (1 - T^{\deg(q_i)})$ gives the number r and the degrees of the w_i.

As there is always one of the w_i with degree 0, we have the following.

3.3. Corollary. $\mathbb{R}[X_1, \ldots, X_6]_{A_5}$ is generated by at most 17 homogeneous polynomials of degree less than or equal to 15.

4. Computing a basis

In order to give a description of the algebra of invariants as proposed earlier we need some elementary facts, first from invariant theory.

Every minimal system of homogeneous generators is irreducible in the sense that every proper subsystem will no more generate the whole algebra. As the algebra of invariants is graded $\mathbb{R}[X_1, \ldots, X_6]_{A_5} =: A = \bigoplus_i A_i$, any irreducible system of homogeneous generators can be computed as follows. We start with $i^{(0)} := 0$, $G^{(0)} := \{\ \}$. If $i^{(n)}$, $G^{(n)}$ are already chosen, then we proceed the following way. If there is an $j > 0$ with $A_j \neq A_j \cap \mathbb{R}[G^{(n)}]$, we put $i^{(n+1)} := \min\{i > i^{(n)} | A_i \neq A_i \cap \mathbb{R}[G^{(n)}]\}$ and enlarge $G^{(n)}$ by a basis of a vector space complement of $A_{i^{(n+1)}}$ relative to $A_{i^{(n+1)}} \cap \mathbb{R}[G^{(n)}]$ and call the new set $G^{(n+1)}$. Otherwise we stop. As A is finitely generated, it is obvious that the procedure will terminate. But it is not evident that the resulting set is already of minimal length. This is assured by the following proposition which also shows that a good polynomial basis can be computed by a similar algorithm.

4.1. Proposition. (i) The notions of an irreducible system of homogeneous generators and of a minimal system of homogeneous generators coincide. Thus, any set of homogeneous generators contains one of minimal length. The degrees of the elements of any minimal system of generators are uniquely determined. More precisely: let A_+ be the ideal generated by the homogeneous elements of positive degree. Then a sequence $(a_1, \ldots, a_r)$ of homogeneous elements in A_+ is a minimal system of homogeneous generators, iff it is a basis of a (graded) vector space complement of A_+^2 in A_+.

(ii) Let $(w_1, \ldots, w_s)$ be a sequence of homogeneous, invariant polynomials. Then

$$\mathbb{R}[X_1, \ldots, X_6]_{A_5} = \bigoplus_{i=1}^{s} \mathbb{R}[q_1, \ldots, q_6] w_i$$

if and only if $(w_1, \ldots, w_s)$ is a minimal system of homogeneous generators of the (graded) $\mathbb{R}[q_1, \ldots, q_6]$-module $\mathbb{R}[X_1, \ldots, X_6]_{A_5}$, i.e. $\mathbb{R}[X_1, \ldots, X_6]_{A_5}$ is equal to

$$\langle\langle w_i \mid i \rangle\rangle \oplus \mathbb{R}[X_1, \ldots, X_6]_{A_5} \{f \in \mathbb{R}[q_1, \ldots, q_6] | f(0) = 0\}$$

as vector space.

Proof. In the case of (i) consider, in a more general context, a finitely generated graded $\mathbb{R}$-algebra A with $A_0 := \mathbb{R}$ and $A_+ \pi \bigoplus_{i>0} A_i$ (cf also Springer 1977, lemma 4.2.6). Hence $\mathbb{R} \simeq A/A_+$. Let π be the canonical homomorphism of graded A-modules $A_+ \to A_+/A_+^2$. As $A_+ \pi(A_+) = 0$, A_+/A_+^2 is, in a natural way, a graded $\mathbb{R}$-vector space. It is easy to see that a sequence (a_i) of homogeneous elements in A_+ is a minimal system of generators, iff $(\pi(a_i))$ is a homogeneous basis of the vector space A_+/A_+^2, iff (a_i) is a homogeneous basis of a vector space complement of A_+^2 in A_+. Thus the theorems of linear algebra can be 'lifted' to A and (i) follows.

Let $R := \mathbb{R}[X_1, \ldots, X_6]$. To prove (ii) we only have to apply the preceding considerations to the R_{S_6}-module R_{A_5}. From this it follows that (w_i) is a minimal system of generators for this module, iff the sequence is a basis of a vector space complement of $R_{A_5}\{f \in R_{S_6} | f(0) = 0\}$ in R_{A_5}. Let $(q_1, \ldots, q_6; z_1, \ldots, z_r)$ be a good polynomial basis. It exists according to Stanley (1979, theorem 3.2). Obviously the (z_i) are a minimal system of generators for the R_{S_6}-module R_{A_5}. Thus, for any minimal system of homogeneous generators $(w_1, \ldots, w_s)$ of this module, we have $r = s$ and, after a suitable rearrangement, $\deg(z_i) = \deg(w_i)$, $i = 1, \ldots, r$. As for arbitrary $i_0 \in \mathbb{N}$ the elements

$(q_1^{i_1}\dots q_6^{i_6}w_i)$, $\Sigma_j\, i_j \deg(q_j)+\deg(w_i)=i_0$, generate $(R_{i_0})_{A_5}$ as vector space, the sequence $(q_1^{i_1}\dots q_6^{i_6}z_i)$, $\Sigma_j\, i_j \deg(q_j)+\deg(z_i)=i_0$, is a basis, and both sets have the same cardinality, $(q_1,\dots,q_6;\, w_1,\dots,w_r)$ is also a good polynomial basis. This proves (ii).

The proposition shows that we can solve (i) and (ii) in § 1 simultaneously. It also proposes a way how this can be done.

Because of theorem 3.2 and proposition 4.1 we only have to consider the spaces $\mathbb{R}[X_1,\dots,X_6]_i$, $i=1,\dots,15$, of homogeneous polynomials of degree i. Now, the action of A_5 on $\mathbb{R}[X_1,\dots,X_6]_i$ also gives a permutation representation, for it permutes the basis $X_1^{i_1}\dots X_6^{i_6}$, $i_1+\dots+i_6=i$.

We list some elementary facts on invariants of permutation representations. Let G be a finite group, D: $\mathrm{G}\to\mathrm{Gl}(V)$ a representation permuting the basis $(v_1,\dots,v_n)$ of a vector space V. It induces a group action of G on the set $\{v_1,\dots,v_n\}$. Let B be the set of its orbits.

4.2. Proposition. The orbit sums

$$\sum_{v\in O} v \qquad O\in B$$

form a basis of the space of G-invariants V_{G}.

If $\mathrm{H}\leqslant\mathrm{G}$ is a subgroup, then the G-orbits in $\{v_1,\dots,v_n\}$ may split. But we have the next proposition.

4.3. Proposition. If $O_{(\mathrm{H})}$ is an H-orbit lying in a G-orbit $O_{(\mathrm{G})}$, then $O_{(\mathrm{G})}$ splits into at most $|\mathrm{G}/\mathrm{H}|$ H-orbits. If L is a transversal of the left cosets G/H, then the set of these orbits is equal to

$$\{\mathrm{D}(g)O_{(\mathrm{H})}|g\in L\}.$$

We recall that each orbit in $\{X_1^{a_1}\dots X_6^{a_6}|\,a_i\}$ and every orbit sum can be parametrised by certain improper partitions into at most six parts (i.e. sixtuples of non-negative integers) (using $[a_1,\dots,a_6]\leftrightarrow X_1^{a_1}\dots X_6^{a_6}$) and that the S_6-invariants can be parametrised by the proper partitions into six parts (i.e. sixtuples of non-negative integers arranged in descending order). Hence, using propositions 4.2 and 4.3, it is easy to describe a basis of each homogeneous component of $\mathbb{R}[X_1,\dots,X_6]_{A_5}$. Computer-aided calculations gave us the following.

4.4. Result. The orbit sums parametrised by

$$\begin{array}{ll}
q_1\leftrightarrow[1,0,0,0,0,0] & q_2\leftrightarrow[2,0,0,0,0,0]\\
q_3\leftrightarrow[3,0,0,0,0,0] & q_4\leftrightarrow[4,0,0,0,0,0]\\
q_5\leftrightarrow[5,0,0,0,0,0] & q_6\leftrightarrow[6,0,0,0,0,0]\\
w_1\leftrightarrow[0,0,0,0,0,0] & w_2\leftrightarrow[1,1,1,0,0,0]\\
w_3\leftrightarrow[2,2,0,1,0,0] & w_4\leftrightarrow[3,1,1,0,1,0]\\
w_5\leftrightarrow[2,2,0,1,0,1] & w_6\leftrightarrow[5,1,1,0,0,0]\\
w_7\leftrightarrow[3,3,0,1,0,1] & w_8\leftrightarrow[2,5,1,1,0,0]\\
w_9\leftrightarrow[3,3,0,3,0,0] & w_{10}\leftrightarrow[5,3,1,1,0,0]\\
w_{11}\leftrightarrow[1,3,2,5,1,0] & w_{12}\leftrightarrow[5,4,3,0,2,1]
\end{array}$$

form a system with properties (ii) of § 1.

The computations also yielded the following.

4.5. Result. The polynomials $q_1, \ldots, q_6, w_2, w_3, w_4, w_6$ of proposition 4.4 form a minimal system of homogeneous generators of the algebra of A_5-invariants. Hence any such system has length 10 with degrees 1, 2, 3, 3, 4, 5, 6, 6, 7.

This gives (i) of § 1.

5. The six-dimensional case

In order to explain the existence of icosahedral symmetry in quasicrystals one has to embed $\mathbb{R}^3$ into $\mathbb{R}^6$ in the following way (see Kramer 1985).

Consider the natural six-dimensional representation ρ of the hyperoctahedral group $\Omega(6) \simeq S_2 \wr S_d$. Then we may identify the full icosahedral group I with a certain subgroup of $\Omega(6)$ (see Kramer 1985). By restricting to I the representation ρ splits into two irreducible components of dimensions three. We have

$$\rho \downarrow \mathrm{I} = ([1^2] \sharp [3, 1^2]^+) \oplus ([1^2] \sharp [3, 1^2]^-).$$

We may now state the problem of describing

$$\mathrm{Sym}(\mathrm{Sym}_2(\mathbb{R}^6))_{\mathrm{I}}$$

the constructions being the same as in the three-dimensional case. As $S_2 \times \{e_{A_5}\}$ acts as $\pm id$ on $\mathbb{R}^6$, hence trivial on $\mathrm{Sym}_2(\mathbb{R}^6)$, we may restrict to

$$\mathrm{Sym}(\mathrm{Sym}_2(\mathbb{R}^6))_{A_5}.$$

The well known representation theory of A_5 gives the following.

5.1. Proposition. The representation of A_5 on $\mathrm{Sym}_2(\mathbb{R}^6)$ splits:

$$2([3, 2] \downarrow A_5 \oplus [1^5] \downarrow A_5) \oplus [3, 2] \downarrow A_5 \oplus [4, 1] \downarrow A_5.$$

It follows immediately that it is a subrepresentation of a representation of $S_6 \times S_6 \times S_6 \times S_5$.

Proof. The character table of A_5 is well known (see, for example, Conway *et al* 1985). The orthogonality relations yield the decomposition into irreducible ones. The second statement is a consequence of the following remarks.

(i) $([3, 2] \oplus [1^5]) \downarrow A_5$ is the permutation representation appearing in the three-dimensional case.

(ii) $[3, 2] \downarrow A_5$ is closely related to the permutation representation in (i). It is equivalent to the representation of case (i) modulo an A_5-stable line.

(iii) $[4, 1] \downarrow A_5$ is the restriction of the well known representation of S_5 which is also a permutation representation modulo an A_5-stable line.

This immediately gives the following.

5.2. Corollary. Let A_5 act on $\tilde{S} := \mathrm{Sym}_2(\mathbb{R}^6)\oplus\mathbb{R}\oplus\mathbb{R}$ such that the corresponding representation is of type $(3([3,2]\oplus[1^5])\oplus([4,1]\oplus[1^5]))\downarrow A_5$ and such that A_5 acts trivially on $\mathbb{R}\oplus\mathbb{R}$. Then there exists a basis of $\tilde{S}$ which is permuted by the elements of A_5.

As $\mathrm{Sym}_2(\mathbb{R}^6)\simeq \tilde{S}/(\mathbb{R}\oplus\mathbb{R})$, there are 23 elements of $\mathrm{Sym}_2(\mathbb{R}^6)$ being permuted by A_5. (They are homomorphic images of the elements of the basis).

5.3. Remark. We cannot expect to find a nice description of the invariants.

(i) Any minimal system of homogeneous generators contains at least 35 elements. The maximal degree of an element of such a system is at least 10 (see also remark 5.6 below).

(ii) If $(q_1, \ldots, q_{21}; w_1, \ldots, w_r)$ is a good polynomial basis, then $\Pi_i \deg(q_i) \geq 2^7 3^7 5^4$. Hence

$$r \geq 2916\,000 \qquad \text{and} \qquad \max\{\deg(w_i)\,|\,i\} \geq 34.$$

(iii) It is also easy to see that (with respect to the decomposition 5.1) there is a good polynomial basis with

$$r = (6!)^4(5!)/60 = 746\,496\,000 \qquad \text{and} \qquad \max\{\deg(w_i)\,|\,i\} = 55.$$

Proof. Let $\mathrm{Sym}_2(\mathbb{R}^6) = \bigoplus_i V_i$ be a decomposition into irreducible invariant subspaces according to proposition 5.1. From result 4.5 and Stanley (1979, ch 4) we conclude that a minimal system of homogeneous generators for $\mathrm{Sym}(V_i)_{A_5}$ consists of 1, 5, 9 elements of degrees (1), (2, 3, 4, 5, 10), (2, 3, 3, 4, 5, 5, 6, 6, 7), if V_i is of type $[1^5]\downarrow A_5$, $[4,1]\downarrow A_5$, $[3,2]\downarrow A_5$, respectively. But $\mathrm{Sym}(\mathrm{Sym}_2(\mathbb{R}^6)) \simeq {}_{A_5}\bigotimes_i \mathrm{Sym}(V_i)$, and we may embed the elements of this system by the natural inclusions $\mathrm{Sym}(V_i)\hookrightarrow \mathrm{Sym}(\mathrm{Sym}_2(\mathbb{R}^6))$. According to Popov (1983, proposition 1.3) we get a sequence that can be enlarged to a minimal system of homogeneous generators of $\mathrm{Sym}(\mathrm{Sym}_2(\mathbb{R}^6))_{A_5}$. This gives (i).

Consider the above decomposition $\bigoplus_i V_i$. From 3.2 and Stanley (1979, ch 4) we know that $\mathrm{Sym}(V_i)_{A_5}$ has a good polynomial basis $(c_1^{(l)}, \ldots, c_{l_l}^{(l)}; d_1^{(l)}, \ldots, d_{r_l}^{(l)})$ with degree sequence $(\deg(c_i^{(l)}))_i$ being (1), (2, 3, 4, 5), (2, 3, 4, 5, 6), if V_l is of type $[1^5]\downarrow A_5$, $[4,1]\downarrow A_5$, $[3,2]\downarrow A_5$, respectively. It is then easy to see that the whole algebra of invariants has a good polynomial basis $(c_1, \ldots, c_{21}; f_1, \ldots, f_r)$, the $(c_i)_i$ consisting of the elements $c_i^{(l)}$ (via the obvious embedding). Thus the degree sequence $\deg(c_i)$ is equal to

$$(1, 1, 2, 3, 4, 5, 2, 3, 4, 5, 6, 2, 3, 4, 5, 6, 2, 3, 4, 5, 6)$$

and (iii) follows from Stanley (1979, corollary 4.3, proposition 4.9).

(ii) is a consequence of the following lemma.

5.4. Lemma. Let $\mathrm{D}\colon \mathrm{G}\to \mathrm{Gl}(V)$ be a representation of a finite group G on a n-dimensional complex vector space, let $v\in\mathbb{C}$ be a primitive uth root of unity and choose $g_0\in \mathrm{G}$ such that v is an eigenvalue of $\mathrm{D}(g_0)$ of multiplicity m.

Then, for every good polynomial basis $(q_1, \ldots, q_n; w_1, \ldots, w_r)$ of $\mathrm{Sym}(V)_{\mathrm{G}}$, $m \leq |\{i\,|\,u \text{ divides } \deg(q_i)\}|$.

Proof. Let $(q_1, \ldots, q_n; w_1, \ldots, w_r)$ be a good polynomial basis for $\mathrm{Sym}(V)_G$. Let χ_ω be an irreducible character of G. From Molien's formula (Stanley 1979, theorem 2.1) we obtain the Poincaré series of the corresponding isotypical component of Sym(V):

$$P_\omega(T) = \frac{\chi_\omega(1)}{|G|} \sum_g \frac{\chi_\omega(g^{-1})}{\det(I - D(g^{-1})T)}.$$

As the irreducible characters and hence also the multiples $((\chi_\omega(1)/|G|)\chi_\omega)$ form a basis of the class functions $Z(G)$, there exist $\lambda_\omega \in \mathbb{C}$ such that

$$\sum_\omega \lambda_\omega \chi_\omega(1)/|G| \chi_\omega$$

is the characteristic function of the conjugacy class of g_0^{-1}. Thus, if C denotes the number of elements of this set, we have

$$\frac{C}{\det(I - D(g_0^{-1})T)} = \sum_\omega \lambda_\omega P_\omega(T).$$

(Note that $\chi_\omega(g^{-1})/\det(I - D(g)T)$ is constant on conjugacy classes.) Hence there is a $P_\omega(T)$ which has a pole of order at least m at $T = v$. But from Stanley (1979, theorem 3.10) we obtain

$$P_\omega(T) = \frac{\Sigma_j T^{\deg(w_j)}}{\Pi_i (1 - T^{\deg(q_i)})}.$$

As $1 - v^{\deg(q_i)} = 0$, iff u divides $\deg(q_i)$, and $1 - T^{\deg(q_i)}$ has only simple zeros, the lemma follows.

We finally examine the Poincaré series.

Molien's formula yields the expansion in the following remark.

5.5. Remark.

$$\begin{aligned} P_{A_5}(T) = {} & 1 + 2T + 10T^2 + 51T^3 + 234T^4 + 1034T^5 + 4206T^6 + 15602T^7 + 53510T^8 \\ & + 170382T^9 + 507422T^{10} + 1423610T^{11} + 3785676T^{12} + 9591910T^{13} \\ & + 23266342T^{14} + 54243496T^{15} + O(T^{16}). \end{aligned}$$

But we get more information by computing a 'multigraded' Poincaré series. Consider the decomposition

$$\mathrm{Sym}_2(\mathbb{R}^6) = V_0 \oplus V_1 \oplus V_2 \oplus V_3 \oplus V_4$$

into invariant subspaces of type $([1^5] \oplus [1^5]) \downarrow A_5, [3,2] \downarrow A_5, [3,2] \downarrow A_5, [3,2] \downarrow A_5, [4,1] \downarrow A_5$, respectively. Then

$$\mathrm{Sym}(\mathrm{Sym}_2(\mathbb{R}^6))_{A_5} \simeq \mathrm{Sym}(V_0)_{A_5} \otimes \left(\bigotimes_{i \geq 1} \mathrm{Sym}(V_i) \right)_{A_5}.$$

$(\bigotimes_{i \geq 1} \mathrm{Sym}(V_i))_{A_5}$ is a $\mathbb{N}^4$-graded algebra in a natural way and we may form the multigraded Poincaré series

$$P_{A_5}(T_1, T_2, T_3, T_4) := \sum_{(i_1, \ldots, i_4)} \dim_{\mathbb{R}} \left(\left(\bigotimes_j \mathrm{Sym}(V_j)_{i_j} \right)_{A_5} \right) T_1^{i_1} \ldots T_4^{i_4}.$$

A multigraded version of Molien's formula yields the next remark.

5.6. Remark.

$$
\begin{aligned}
P_{A_5}(T_1, T_2, T_3, T_4)
&= 1+(T_4^2+T_2T_3+T_1T_3+T_3^2+T_1T_2+T_2^2+T_1^2)\\
&\quad+(2T_1^3+T_4^3+2T_2^3+2T_3^3+2T_1T_3T_4+2T_1T_2^2+2T_1^2T_2\\
&\quad+2T_1T_2T_3+T_3T_4^2+T_2T_4^2+T_1T_4^2+T_3^2T_4+T_2^2T_4\\
&\quad+T_1^2T_4+2T_2T_3^2+2T_1T_3^2+2T_2^2T_3+2T_1^2T_3+2T_1T_2T_4+2T_2T_3T_4)\\
&\quad+(7T_1T_2^2T_3+5T_1^2T_2T_4+5T_2T_3^2T_4+5T_2T_3T_4^2+5T_1^2T_3T_4\\
&\quad+5T_1T_3^2T_4+2T_1^4+2T_2^4+3T_1T_2^3+6T_1^2T_2^2+3T_1^3T_2\\
&\quad+2T_3^4+2T_4^4+T_3T_4^3+T_2T_4^3+T_1T_4^3+3T_1T_3^3+3T_2T_3^3\\
&\quad+3T_1^3T_3+3T_2^3T_3+6T_2^2T_3^2+6T_1^2T_3^2+4T_3^2T_4^2\\
&\quad+4T_2^2T_4^2+4T_1^2T_4^2+7T_1T_2T_3^2+7T_1^2T_2T_3+5T_1T_3T_4^2\\
&\quad+5T_1T_2T_4^2+5T_2^2T_3T_4+5T_1T_2^2T_4+3T_3^3T_4+3T_2^3T_4\\
&\quad+3T_1^3T_4+8T_1T_2T_3T_4)+\text{terms of higher order.}
\end{aligned}
$$

Specialising to $T_1=T_2=T_3=T_4=T$, i.e. to the gradation given by total degree, we obtain a Poincaré series whose expansion is

$$1+7T^2+33T^3+142T^4+617T^5+2372T^6+8224T^7+\mathrm{O}(T^8).$$

From this we immediately conclude that a minimal system of homogeneous generators for $\mathrm{Sym}(\mathrm{Sym}_2(\mathbb{R}^6))_{A_5}$ consists of more than 2307 elements.

Note. Most of our calculations were made on a HP 9500 using Pascal programs and the MAPLE programming-system (Char *et al* 1985).

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